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Superconformal field theories for compact Manifolds with Spin(7) holonomy

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Abstract: We present a construction of superconformal field theories for manifolds with Spin(7) holonomy. Geometrically these models correspond to the realization of Spin(7) manifolds as anti-holomorphic quotients of Calabi-Yau fourfolds. Describing the fourfolds as Gepner models and requiring anomaly cancellation we determine the resulting Betti numbers of the Spin(7) superconformal field theory. As in the G_2 case, we find that the Gepner model and the geometric result disagree.

Keywords: Superstring Vacua, Conformal Field Models in String Theory, Conformal and W Symmetry.

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Contents

1. Introduction

Mainly during the last year we have seen an intensified interest into the features of M-theory respectively string theory on manifolds of exceptional holonomy [1]–[21]. In particular, transitions in the geometry of M-theory on non-compact G_2 manifolds give rise to non-trivial dualities in $\mathcal{N}=1$ supersymmetric gauge theories [22]–[27].

Besides all these successes it is also of importance to study M-theory and string theory on compact manifolds of G_2 and Spin(7) holonomy. Starting with the early work of Shatashvili and Vafa [28], recently there have been a couple of papers about the features of the superconformal fields theories describing strings moving on manifolds of exceptional holonomy [29]–[34].

In [32] superconformal fields theories (SCFTs) have been constructed describing strings on G_2 manifolds of the form $(CY_3 \times S^1)/\mathbb{Z}_2$, where the Calabi-Yau threefold was realized as a Gepner model. Discussing three examples in detail it was found that the $\mathbb{Z}_2$ twisted sectors in the SCFT were different from what one expects in the geometric large volume phase. This was explained by the appearance of a discrete NS-NS two-form flux in the Gepner model.[1]

In this paper we generalize the construction to the case of manifolds with Spin(7) holonomy. Starting with the Gepner model for a type-IIB string on a Calabi-Yau fourfold, we show that, dividing by an anti-holomorphic $\mathbb{Z}_2$ symmetry, one gets a SCFT satisfying all the requirements for a compactification on a Spin(7) manifolds. Requiring cancellation of the gravitational anomaly in the low energy two-dimensional theory we will derive the Betti numbers of all Spin(7) manifolds, realized as anti-holomorphic quotients of Gepner

[1] The partition functions for all Gepner models were constructed very recently in [34].

models, in terms of the Hodge numbers of the underlying Calabi-Yau fourfold. Similar to the G_2 case, the SCFT results disagree in all cases with the supergravity expectation. It was suggested that the resolution of this puzzle is that the SCFT and the supergravity limit lie on disconnected branches in the stringy moduli space.

This paper is organized as follows. In section 2 we discuss Gepner models with $c = 12$ describing certain points in the moduli space of Calabi-Yau fourfolds. In particular, we compute from the SCFT the massless spectrum and show that it automatically satisfies anomaly cancellation. In section 3 we divide these Gepner models by an anti-holomorphic involution, determine the $\mathbb{Z}_2$ twisted sector partition function for the $(2)^2(6)^4$ Gepner model in detail and compute the massless spectrum for all possible Gepner models by employing anomaly cancellation. We also compare the SCFT result to the geometric spectrum and give arguments for their discrepancy. Finally, section 4 contains our conclusions.

2. Gepner models for Calabi-Yau fourfolds

In this paper we use the same notation as in [32, 35] and for a general introduction to Gepner models we refer the reader to the original literature [36, 37].

It is well known [38, 39] that purely geometric type-IIA compactifications on Calabi-Yau fourfolds are destabilized by the one loop term

$$-\int B \wedge X_8(R)\,, \tag{2.1}$$

where B denotes the NS-NS two-form and $X_8(R)$ a special quartic polynom in the curvature of the fourfold. For non-vanishing $\chi/24 = \int_{CY_4} X_8$ the term (2.1) generates a tadpole for the NS-NS two-form B. Thus, for getting stable vacua one has to either introduce fundamental strings in the background or turn on some fluxes on the Calabi-Yau fourfolds. In order to avoid this, we will consider only type-IIB compactifications in the following. As shown in [40], such models are consistent and in particular automatically lead to anomaly free models in two space-time dimensions.

Some new features appear in two-dimensions: first there exist Majorana-Weyl spinors which are either right-moving (chiral) or left-moving (antichiral). Similarly one has to distinguish between chiral and anti-chiral bosons as well. Second, in light-cone gauge it is not immediately clear how one can determine the two-dimensional massless spectrum directly from the string theoretic partition function. However, by formally introducing characters of SO(0) we will argue that it is indeed still possible.

Compactifying the type-IIB string on a Calabi-Yau fourfold yields $\mathcal{N} = (0, 4)$ supersymmetry in two dimensions. The massless spectrum consists of certain numbers of chiral supermultiplets $(4\phi^+, 4\psi^+)$, anti-chiral bosons ϕ^- and anti-chiral fermions ψ^-.

In the remainder of this section we will discuss how these multiplicities can be determined from the Gepner model partition function. The Gepner model is constructed from a tensor product of N unitary models of the $\mathcal{N} = 2$ Virasoro algebra with central charge $c = 12$. In the bosonic string this internal sector is combined with the $\mathrm{SO}(8) \times E_8$ current algebra at level $k = 1$. The conformal dimensions and the $U(1)$ charges of the four highest weight representations of $\mathrm{SO}(4n)$ are displayed in table 1.

reps.	h	q
O_{2n}	0	0
V_{2n}	1/2	1
S_{2n}	$n/4$	0
C_{2n}	$n/4$	1

Table 1: SO($4n$) representations.

	O	V	S	C
O	O	V	S	C
V	V	O	C	S
S	S	C	O	V
C	C	S	V	O

Table 2: SO($4n$) fusion rules.

Moreover, from the modular S-matrix

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{2.2}$$

one derives the fusion rules shown in table 2.

Note in particular, that the fusion of the spinor respectively antispinor representation with itself gives the singlet representation. The Gepner model is constructed by starting with the diagonal partition function and then taking orbits under both the simple current [41, 42] implementing the GSO projection

$$J_{GSO} = (0,1,1)^N \otimes (S_8) \tag{2.3}$$

and the N simple currents

$$J_a = \prod_{i=1}^{a-1}(0,0,0) \otimes (0,0,2) \otimes \prod_{i=a+1}^{N}(0,0,0) \otimes (V_8) \tag{2.4}$$

ensuring that from the individual factor theories only NS respectively R sector states are combined. After formally applying the bosonic string map exchanging SO(8) $\times$ E_8 and SO(0) representations in the following way

$$(O_8, V_8, S_8, C_8) \rightarrow (V_0, O_0, -C_0, -S_0) \tag{2.5}$$

one gets the partition function of the $c = 12$ Gepner model.

A Calabi-Yau fourfold has four non-trivial Hodge numbers h_{31}, h_{21}, h_{11} and h_{22} satisfying the relation

$$h_{22} = 44 + 4\,h_{11} + 4\,h_{31} - 2\,h_{21}. \tag{2.6}$$

Can one detect these numbers in the chiral ring of the Gepner model? In table 3 we summarize how the massless orbits in the Gepner model are connected to the Hodge numbers of the Calabi-Yau manifold.

Note, that the equation (2.6) is automatically satisfied in the Gepner model. Since the character of the vector representation of SO(0) is formally vanishing (a vector in two dimensions has no on-shell degrees of freedom), there do not appear any massless NS states

Hodge	left	right
h_{31}	$\left(\frac{1}{2},\mp 1\right)_{NS}\otimes O_0-\left(\frac{1}{2},\pm 1\right)_R\otimes S_0$	$\left(\frac{1}{2},\mp 1\right)_{NS}\otimes O_0-\left(\frac{1}{2},\pm 1\right)_R\otimes S_0$
h_{11}	$\left(\frac{1}{2},\mp 1\right)_{NS}\otimes O_0-\left(\frac{1}{2},\pm 1\right)_R\otimes S_0$	$\left(\frac{1}{2},\pm 1\right)_{NS}\otimes O_0-\left(\frac{1}{2},\mp 1\right)_R\otimes S_0$
h_{21}	$\left(\frac{1}{2},\mp 1\right)_{NS}\otimes O_0-\left(\frac{1}{2},\pm 1\right)_R\otimes S_0$	$(1,\mp 2)_{NS}\otimes V_0-\left(\frac{1}{2},0\right)_R\otimes C_0$
h_{22}	$(1,\mp 2)_{NS}\otimes V_0-\left(\frac{1}{2},0\right)_R\otimes C_0$	$(1,\mp 2)_{NS}\otimes V_0-\left(\frac{1}{2},0\right)_R\otimes C_0$

Table 3: Massless orbits.

sector	left-right SO(0) comb.	massless state
NS-NS	$(O_0)_L\times(O_0)_R$	1 non-chiral boson
R-R	$(S_0)_L\times(S_0)_R$	1 (anti)-chiral boson
R-R	$(C_0)_L\times(C_0)_R$	1 (anti)-chiral boson
R-R	$(S_0)_{L,R}\times(C_0)_{R,L}$	–
NS-R	$(O_0)_{L,R}\times(S_0)_{R,L}$	1 chiral fermion
NS-R	$(O_0)_{L,R}\times(C_0)_{R,L}$	1 anti-chiral fermion
NS-R	$(V_0)_{L,R}\times(S_0,C_0)_{R,L}$	–

Table 4: Massless states.

in the orbits related to h_{21} and h_{22}. Beside the matter multiplets there also appear R-R massless states in the vacuum orbit

$$\begin{aligned}&\left[(0,0)_{NS}\otimes V_0-\left(\frac{1}{2},2\right)_R\otimes C_0-\left(\frac{1}{2},-2\right)_R\otimes C_0\right]_L\times\\&\times\left[(0,0)_{NS}\otimes V_0-\left(\frac{1}{2},2\right)_R\otimes C_0-\left(\frac{1}{2},-2\right)_R\otimes C_0\right]_R.\end{aligned}\tag{2.7}$$

The type-IIB massless spectrum can now be computed by applying the rules shown in table 4.

Neglecting the dilaton which is part of the supergravity multiplet and taking the four (anti-)chiral bosons from the vacuum orbit into account the total number of (anti-)chiral bosons is

$$n_{\phi^+}+n_{\phi^-}=6\,h_{31}+6\,h_{11}\,+h_{22}+4\,.\tag{2.8}$$

For the number of chiral respectively antichiral fermions one obtains

$$\begin{aligned}n_{\psi^+}&=4\,(h_{31}+h_{11})\\n_{\psi^-}&=4\,h_{21}\,.\end{aligned}\tag{2.9}$$

Since 4 chiral fermions and 4 chiral bosons form one $\mathcal{N}=(0,4)$ supermultiplet, one can deduce the number of chiral and antichiral bosons separately

$$\begin{aligned}n_{\phi^+}&=4\,(h_{31}+h_{11})\\n_{\phi^-}&=6\,h_{31}+6\,h_{11}-2\,h_{21}+48\,,\end{aligned}\tag{2.10}$$

where we have used (2.6). The gravitational anomaly for $\mathcal{N}=(0,4)$ supergravity theory in two dimensions is proportional to

$$4+\frac{n_{\phi^+}-n_{\phi^-}}{12}+\frac{n_{\psi^+}-n_{\psi^-}}{24}\,, \tag{2.11}$$

where the first term comes from the gravitinos. Indeed, for the massless spectrum shown in (2.9) and (2.10) the anomaly vanishes as it should be for a modular invariant stringy partition function.

3. Gepner models for Spin(7) manifolds

Given a Gepner model with $c=12$, we now divide by the anti-holomorphic involution σ^* which geometrically acts as complex conjugation. Therefore, the supersymmetry on the world-sheet is broken to $\mathcal{N}=1$. In complete analogy to [32], in the Gepner model σ^* acts as charge conjugation in each individual $\mathcal{N}=2$ tensor factor. Thus, in the $\sigma^* \underset{1}{\Box}$ sector of the orbifold partition function only uncharged states can contribute. As opposed to the G_2 case, for all levels k_i even there do not only exist uncharged states in the NS-NS sector but also in the R-R sector. Let us discuss this case first.

3.1 All levels even

Using the action of the anti-holomorphic involution on the $\mathcal{N}=2$ unitary models with level $k=1,2,3,6$ as derived in [32], we have computed the orbifold of the $(2)^2(6)^4$ Gepner model in detail. Geometrically, the $(2)^2(6)^4$ Gepner model corresponds to the Calabi-Yau fourfold $\mathbb{P}_{1,1,1,1,2,2}[8]$ with Hodge numbers $h_{31}=443$, $h_{11}=1$, $h_{21}=0$ and $h_{22}=1820$.

The computation for the partition function is absolutely analogous to the G_2 examples discussed in [32]. We merely use this concrete example as a guiding model for the general results presented in the following.

The uncharged states in the NS-NS sector, which are contained in the highest weight representations (HWR) $(l,m,s)=(l,0,0)$ of the $\mathcal{N}=2$ unitary models contribute to the $\sigma^* \underset{1}{\Box}$ sector of the $\mathbb{Z}_2$ orbifold in the same way as in [32]. However, as the $(2)^2(6)^4$ example

$$\begin{aligned}\sigma^* \underset{1}{\Box}^{NS} = \frac{1}{2}\,|V_0|^2 \sum_{i,j=0}^{1}\ \sum_{k,l,m,n=0}^{3} \Big| \psi^{2i}(\sigma^*)\,\psi^{2j}(\sigma^*)\Big(\chi^{2k}(\sigma^*)\,\chi^{2l}(\sigma^*)\,\chi^{2m}(\sigma^*)\,\chi^{2n}(\sigma^*)\,+ \\ +\,\chi^{6-2k}(\sigma^*)\,\chi^{6-2l}(\sigma^*)\,\chi^{6-2m}(\sigma^*)\,\chi^{6-2n}(\sigma^*)\Big)\Big|^2\end{aligned} \tag{3.1}$$

reveals in detail the NS-NS sector alone does only yield half-integer coefficients in the twisted sector partition function, which would not allow the interpretation as a trace over states in a Hilbert space. In (3.1) we have used the notation from [32].

Since the state $\prod_i(\frac{k_i}{2},\frac{k_i}{2}+1,1)\otimes(C_0)$ in the Ramond sector is uncharged, the R-R sector also contributes to $\sigma^* \underset{1}{\Box}$. Taking this into account, for the $(2)^2(6)^4$ Gepner model one gets

$$\sigma^* \underset{1}{\Box}^{R} = 2\,|C_0|^2\,\left|\left(\psi^1(\sigma^*)\right)^2\left(\chi^3(\sigma^*)\right)^4\right|^2, \tag{3.2}$$

JHEP12(2001)013

where the extra factor of two signals that the relevant orbit has half the length of the standard orbit. The $k=2$ and $k=6$ characters in (3.2) are

$$\begin{aligned}\psi^1(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\sqrt{\frac{\theta_2}{2\eta}} \\ \chi^3(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\,\frac{1}{2}\left(\chi^R_{\frac{3}{32}}-\chi^R_{\frac{65}{32}}\right),\end{aligned} \tag{3.3}$$

using the fact that the $\mathcal{N}=2$ unitary model at level $k=6$ can be expressed as the unitary model of the $\mathcal{N}=1$ super Virasoro algebra at level $m=6$.

A modular S-transformation to the σ^* twisted sector yields

$$1\,\underset{\sigma^*}{\square} = \left|\sqrt{\frac{\eta}{\theta_4}}\right|^6 \frac{1}{2}\Big[I_1(q,\overline{q})\,|O_0+V_0-S_0-C_0|^2 + I_2(q,\overline{q})\,|O_0-V_0-S_0+C_0|^2\Big], \tag{3.4}$$

where the first term is the modular transform of the NS-NS sector contribution to ${}_{\sigma^*}\underset{1}{\square}$ and the second term the R-R sector contribution. The series I_1 and I_2 start like

$$\begin{aligned}I_1(q,\overline{q}) &= \left|\psi_0^2\left(\chi^{NS}_{\frac{1}{32}}\right)^4\right| + \cdots \\ I_2(q,\overline{q}) &= \left|\psi_0^2\left(\chi^{\widetilde{NS}}_{\frac{1}{32}}\right)^4\right| + \cdots .\end{aligned} \tag{3.5}$$

The complete twisted sector (3.4) has indeed integer coefficient and therefore allows the interpretation as a trace. Moreover, the twisted sector satisfies level matching and is free of tachyons. It turns out that the ground state in the twisted sector is massless.

However, before determining the massless spectrum in the twisted sector let us compute the massless spectrum from the untwisted sector not only for the $(2)^2(6)^4$ Gepner model but quite generally for every Gepner model with even levels. First, since σ^* acts as charge conjugation, the vacuum orbit tells us that supersymmetry is broken to $\mathcal{N}=(0,2)$.

Moreover, neglecting the dilaton we get two (anti-)chiral bosons from the vacuum orbit. The first and second class of orbits in table 3 contribute $3h_{31}+3h_{11}$ (anti-)chiral bosons, while as in the fourfold case the third class of orbits does not lead to any boson at all. For the states in the fourth class of orbits, one has to take into account that (only) the left-right combination

$$\left[\prod_i\left(\frac{k_i}{2},\frac{k_i}{2}+1,1\right)\otimes(C_0)\right]_L \times \left[\prod_i\left(\frac{k_i}{2},\frac{k_i}{2}+1,1\right)\otimes(C_0)\right]_R \tag{3.6}$$

is invariant under σ^*. As mentioned above, this orbit has half the length of the standard orbit and therefore we find that the total number of (anti-)chiral bosons arising in the fourth class of orbits is $h_{22}/2+1$. The overall number of (anti-)chiral bosons is therefore

$$n_{\phi^+}+n_{\phi^-} = 3\,h_{31}+3\,h_{11}+\frac{h_{22}}{2}+3\,. \tag{3.7}$$

Since precisely half of the fermions are projected out we get

$$\begin{aligned}n_{\psi^+} &= 2\,(h_{31}+h_{11}) \\ n_{\psi^-} &= 2\,h_{21}\,,\end{aligned} \tag{3.8}$$

so that we can deduce the number of chiral and antichiral bosons separately

$$\begin{aligned} n_{\phi^+} &= 2\,(h_{31}+h_{11}) \\ n_{\phi^-} &= 3\,h_{31}+3\,h_{11}-h_{21}+25\,. \end{aligned} \tag{3.9}$$

Note that for this spectrum the gravitational anomaly

$$I = 2 + \frac{n_{\phi^+}-n_{\phi^-}}{12} + \frac{n_{\psi^+}-n_{\psi^-}}{24} \tag{3.10}$$

does not vanish but gives $I = -1/12$. This should be canceled by the contribution from the massless modes in the $\mathbb{Z}_2$ twisted sector. The partition function (3.4) derived for the $(2)^2(6)^4$ Gepner model gives rise to the massless modes

$$\begin{aligned} n^{tw}_{\phi^+} &= 2\,, \qquad n^{tw}_{\phi^-} = 2 \\ n^{tw}_{\psi^+} &= 2\,, \qquad n^{tw}_{\psi^-} = 0 \end{aligned} \tag{3.11}$$

which precisely contribute the missing states to cancel the gravitational anomaly.

Even though we have explicitly computed the twisted sector partition function only for the $(2)^2\,(6)^4$ Gepner model, turning the argument around, anomaly cancellation tells us that for all Gepner models with only even levels there must be the same contribution (3.11) from the $\mathbb{Z}_2$ twisted sector.[2]

By dimensional reduction of type-IIB string theory on a Spin(7) manifold we expect the number of chiral respectively anti-chiral two dimensional bosons and fermions to be

$$\begin{aligned} n^{geo}_{\phi^+} &= 2\,b_2+2\,b_4^-+2 \\ n^{geo}_{\phi^-} &= 2\,b_2+b_4+2 \\ n^{geo}_{\psi^-} &= 2\,b_3, \end{aligned} \tag{3.12}$$

where b_n denote the Betti numbers of the Spin(7) manifold: there are b_4^-+1 geometric moduli giving rise to the same number of non-chiral bosons. The NS-NS and R-R two-forms yield $2b_2$ non-chiral bosons. The R-R zero-form gives another non-chiral boson and the self-dual four-form gives rise to b_4^- chiral bosons and b_4^+ anti-chiral bosons.

Now we can try to compute the Betti numbers of the Spin(7) SCFT model in terms of the Hodge numbers of the Calabi-Yau fourfold we started with. However, from the SCFT data one can not uniquely determine the number of σ^* invariant 2-cycles, so that we can only say that $b_2 = h_{11}+1-l$ with $0 \le l \le h_{11}+1$. Here we have also taken the possible contribution from the $\mathbb{Z}_2$ twisted sector (3.11) into account. Using $b_3 = h_{21}$ we get for the number of self-dual and anti self-dual 4-cycles

$$\begin{aligned} b_2 &= h_{11}-l+1 \\ b_3 &= h_{21} \\ b_4^- &= h_{31}+l-1 \\ b_4^+ &= 2\,h_{31}+h_{11}-h_{21}+l+24\,. \end{aligned} \tag{3.13}$$

[2]While this paper was in preparation we received [34], which showed quite generally that for the G_2 Gepner models with only even levels one always finds 2 chiral states from the $\mathbb{Z}_2$ twisted sector. Apparently, something very similar happens in the Spin(7) case.

The resulting Euler number is

$$\chi = 2 + 2\,b_2 - 2\,b_3 + b_4 = 3(h_{31} - h_{21} + h_{11} + 9) \tag{3.14}$$

which is in particular independent of l. The existence of one covariantly constant spinor on a Spin(7) manifold implies via the index theorem a relation between the Betti numbers

$$24 = b_3 - b_2 - 2\,b_4^- + b_4^+ - 1\,. \tag{3.15}$$

We consider it as a nice check of our computation that the Betti numbers derived from the SCFT analysis indeed automatically satisfy this condition independently of l. Thus, up to the parameter l the SCFT determines the Betti numbers completely, which was though expected from the analysis in [28]. Since the SCFT only fixes the sums $b_2 + b_4^-$ and $b_2 + b_4^+$, it was conjectured in [28] that this reflects a generalized mirror symmetry for Spin(7) manifolds. All geometric target spaces with Betti numbers as in (3.13), for each l lead to the same SCFT and can therefore not be distinguished by strings moving in these backgrounds.

3.2 The large radius geometry

The geometric phase corresponding to a Gepner model with only even levels is a Calabi-Yau hypersurface of the form

$$\sum_{i=1}^{6} z_i^{k_i+2} = 0 \tag{3.16}$$

in an appropriate weighted projective space. Thus the $\mathbb{Z}_2$ acts freely on the fourfold since there are no real solutions. Furthermore the involution is antiholomorphic and therefore inverts the Kähler class arising from the ambient space. So there are $h_{11} - l$ invariant 2–cycles with $0 < l \leq h_{11}$ allowing us to determine the remaining Betti numbers. Obviously $b_0 = 1$, $b_1 = 0$, $b_2 = h_{11} - l$. Then the involution exchanges $H^{2,1} \leftrightarrow H^{1,2}$ and therefore $b_3 = h_{21}$.

It is not obvious how the involution acts on $H^{2,2}$, but we can determine this from the Euler number:

$$\begin{aligned} 2 + 2\,b_2 - 2\,b_3 + b_4 \;=\; \chi(Y/\mathbb{Z}_2) = \frac{1}{2}\chi(Y) &= \frac{2 + 2\,h_{11} - 4\,h_{21} + 2 + 2\,h_{31} + h_{22}}{2} \\ \Rightarrow b_4 = 2l - h_{11} + h_{31} + \frac{h_{22}}{2} &= 22 + 2l + h_{11} + 3\,h_{31} - h_{21} \end{aligned} \tag{3.17}$$

so there are $h_{22}/2 + 2l - h_{11} - 1$ classes invariant in $H^{2,2}(Y)$. Finally we determine the refined Betti numbers $b_4^{\pm}$ from $\hat{A}(Y/\mathbb{Z}_2) = 1$, that is from eq. (3.15). We find

$$\begin{aligned} b_4^- &= h_{31} + l - 1 \\ b_4^+ &= 2\,h_{31} + h_{11} - h_{21} + l + 23\,. \end{aligned} \tag{3.18}$$

The quotient $Y/\mathbb{Z}_2$ of course has $\pi_1 = \mathbb{Z}_2$ and therefore not the full Spin(7) holonomy, but rather $\mathrm{Hol}(Y/\mathbb{Z}_2) = \mathbb{Z}_2 \ltimes SU(4)$. However there is one constant spinor and this suffices for our purposes.

3.3 SCFT vs. geometry

Similar to the G_2 Gepner models the SCFT result disagrees with the large radius computation: from the analysis of the geometric phase one would not expect any massless states in the twisted sector, and indeed the computation of the Betti numbers misses these degrees of freedom. We expect the resolution to this puzzle to be similar to the G_2 Gepner models. There we argued that the moduli coming from the dimensional reduction of the NS-NS 2-form are frozen in the G_2 manifold, so that one can not continuously go from the Gepner model to the geometric phase.

Here if a two-cycle is anti-invariant under the anti-holomorphic involution, which is always the case for the Kähler class inherited from the ambient space, the B field can still take the two discrete values $B = 0$ and $B = 1/2$. Since generally the Gepner model is expected to correspond to a point in the moduli where $B = 1/2$, the SCFT model and the supergravity limit lie on disconnected branches of the moduli space.

3.4 At least one level odd

If at least one level of the $c = 12$ Gepner model is odd, there are no uncharged states in the R-R sector. Therefore, the massless spectrum in the untwisted sector is precisely half the spectrum of the fourfold Gepner model. The number of (anti-)chiral fermions and bosons turns out to be

$$\begin{aligned} n_{\psi^+} &= 2\,(h_{31} + h_{11}) \\ n_{\psi^-} &= 2\,h_{21} \\ n_{\phi^+} &= 2\,(h_{31} + h_{11}) \\ n_{\phi^-} &= 3\,h_{31} + 3\,h_{11} - h_{21} + 24\,, \end{aligned} \tag{3.19}$$

which by themselves cancel the gravitational anomaly. Therefore, we do not expect any further massless states in the twisted sector which is in accord with the analogous result for the G_2 Gepner models [32, 33, 34]. Comparing the SCFT spectrum (3.19) with the expected result from dimensional reduction (3.12), one can again deduce the Betti numbers of the Spin(7) SCFT

$$\begin{aligned} b_2 &= h_{11} - l \\ b_3 &= h_{21} \\ b_4^- &= h_{31} + l - 1 \\ b_4^+ &= 2\,h_{31} + h_{11} - h_{21} + l + 23\,. \end{aligned} \tag{3.20}$$

Note, that these Betti-numbers are slightly different from the ones in (3.13), but agree with the geometric expressions (3.18) derived for a free orbifold action. The resulting Euler number is

$$\chi = 2 + 2\,b_2 - 2\,b_3 + b_4 = 3(h_{31} - h_{21} + h_{11} + 8)\,. \tag{3.21}$$

In this case, in the expression for the hypersurface

$$\sum_{i=1}^{6} z_i^{k_i+2} = 0 \tag{3.22}$$

at least one exponent is odd and one would expect non-trivial fixed loci under complex conjugation. These would lead to further massless states in the twisted sector, so that also here the SCFT result and the geometric result disagree.

4. Conclusions

In this paper we have generalized the construction of exactly solvable SCFTs to the case of Spin(7) manifolds. Starting with a Gepner model with $c = 12$ describing certain points in the moduli space of string compactifications on Calabi-Yau fourfolds, we divided by charge conjugation to get a Spin(7) manifold. The massless type-IIB spectra we got satisfy all the conditions we expect for Spin(7) compactifications. Employing the cancellation of the gravitational anomaly in two dimensions, it was possible to compute for all Gepner models the resulting Betti numbers of the Spin(7) manifold in terms of the Hodge numbers of the Calabi-Yau fourfold. One might speculate that these formulae are not only restricted to Gepner models but might be true even for more general hypersurfaces in weighted projective spaces respectively for more general orbifolds of $c = 12$ Landau-Ginzburg models [43].

Similar to the G_2 case the twisted sectors for the SCFT and the supergravity computations turned out to be different, reflecting the fact that both models lie on separate branches of the Spin(7) moduli space.

To establish this picture of the moduli space we argue that the large volume and the Gepner point carry a distinct discrete modulus: the value of the B–field. Although this does not explain the *reason* for the different number of degrees of freedom (we expect it to be rather a stringy effect at small radius), this does show that these two limits are on different components of the moduli space.

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